P.205 : 4

For $0 < |z| < 1$:

$$f(z) = \frac{1}{z^2} \cdot \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^n + \frac{1}{z} + \frac{1}{z^2}$$

For $1 < |z| < \infty$:

$$f(z) = -\frac{1}{z^3} \cdot \frac{1}{1-\frac{1}{z}} = -\frac{1}{z^3} \cdot \sum_{n=0}^{\infty} \frac{1}{z^n} = -\sum_{n=3}^{\infty} \frac{1}{z^n}$$

5. For $z \in D_1$:

$$f(z) = -\frac{1}{1-z} + \frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty} z^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \cdots$$

For $z \in D_2$:

$$f(z) = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} + \frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = \frac{1}{z} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \cdots$$

For $z \in D_3$:

$$f(z) = \frac{1}{z} \left(\frac{1}{1-\frac{1}{z}} - \frac{1}{1-\frac{2}{z}} \right) = \frac{1}{z} \left(\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n \right)$$

$$= \cdots$$

7. (a) $\left|\frac{a}{z}\right| < 1 \Rightarrow \frac{a}{z-a} = \frac{a}{z} \cdot \frac{1}{1-\frac{a}{z}}$

$$= \frac{a}{z} \cdot \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \cdots$$

(b) $$\frac{a}{z-a} = \frac{a}{\cos\theta + i\sin\theta - a} = \frac{a\cos\theta - a^2 - i\,a\sin\theta}{1 - 2a\cos\theta + a^2}$$

$$\frac{a^n}{z^n} = a^n \cdot e^{-in\theta} = a^n(\cos n\theta - i\sin n\theta)$$

Compare Re & Im parts and we're done.

P.218: 1

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2} = \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right) = \sum_{n=0}^{\infty} (n+1) z^n$$

$$\frac{2}{(1-z)^3} = \frac{d}{dz}\left(\frac{1}{(1-z)^2}\right) = \frac{d}{dz}\left(\sum_{n=0}^{\infty} (n+1) z^n\right) = \sum_{n=0}^{\infty} (n+1)(n+2) z^n$$

2. $$\frac{1}{\left(1-\frac{1}{1-z}\right)^2} = \frac{(1-z)^2}{z^2} = \sum_{n=0}^{\infty} (n+1)\left(\frac{1}{1-z}\right)^n$$

$$\Rightarrow \frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{(z-1)^{n+2}} = \sum_{n=2}^{\infty} \cdots$$

5. $$f(z) = \frac{\cos z}{\pi\left(z-\frac{\pi}{2}\right)} - \frac{\cos z}{\pi\left(z+\frac{\pi}{2}\right)} = T(z), \quad z \neq \pm\frac{\pi}{2}$$

At $z = \frac{\pi}{2}$:

$$T(z) = \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(z-\frac{\pi}{2}\right)^{2n} - \frac{\cos z}{\pi\left(z+\frac{\pi}{2}\right)}$$

$$T\left(\frac{\pi}{2}\right) = -\frac{1}{\pi} = f\left(\frac{\pi}{2}\right)$$

At $z = -\frac{\pi}{2}$:

$$T(z) = \frac{\cos z}{\pi\left(z-\frac{\pi}{2}\right)} - \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(z+\frac{\pi}{2}\right)^{2n}$$

$$T\left(-\frac{\pi}{2}\right) = -\frac{1}{\pi} = f\left(-\frac{\pi}{2}\right)$$

This proves that f is entire.

P. 242 : 1

(a) $z \cdot e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^{n-1}}$ The principal part is $\sum_{n=2}^{\infty} \frac{1}{n!\, z^{n-1}}$

0 is an essential singular point.

(b) $\frac{z^2}{1+z} = z - 1 + \frac{1}{1+z}$. The principal part is $\frac{1}{1+z}$

-1 is a pole.

(c) $\frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}$. Trivial principal part

0 is a removable singular point.

(d) $\frac{\cos z}{z} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n-1}}{(2n)!}$. The principal part is $\frac{1}{z}$

0 is a pole.

(e) Principal part : $\frac{-1}{(z-2)^3}$; 2 is a pole.

2. (a) $\frac{1 - \cosh z}{z^3} = \frac{1}{z^3}\left(1 - \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\right) = -\frac{1}{2z} - \frac{z}{4!} - \frac{z^3}{6!} - \cdots$

0 is a ~~pte~~ pole. $m = 1$, $B = -\frac{1}{2}$

(b) $\frac{1 - e^{2z}}{z^4} = \frac{1}{z^4}\left(1 - \sum_{n=0}^{\infty} \frac{(2z)^n}{n!}\right) = -\frac{2}{z^3} - \frac{2}{z^2} - \frac{4}{3z} - \frac{2}{3} - \cdots$

0 is a pole. $m = 1$, $B = -\frac{4}{3}$

(c) $\frac{e^{2z}}{(z-1)^2} = \frac{1}{(z-1)^2} \cdot e^2 \cdot \sum_{n=0}^{\infty} \frac{2^n}{n!} (z-1)^n$

$$= \frac{e^2}{(z-1)^2} + \frac{2e^2}{z-1} + 2e^2 + \frac{4}{3} e^2 (z-1)$$

~~Q~~ 1 is a pole. $m = 2$, $B = 2e^2$

P. 242: 4

$\phi(z)$ is analytic at ai, so we can write it as Taylor series: $\phi(z) = \cancel{\phi(ai)} \sum_{n=0}^{\infty} \frac{\phi^{(n)}(ai)}{n!}(z-ai)^n$

Substitude $\phi(z)$ in $f(z) = \frac{\phi(z)}{(z-ai)^3}$

Then we obtain the principal part of $f(z)$ easily.

Calculate $\phi'(z)$ and $\phi''(z)$, and the last step is done.

P. 246: 1

(a) $f(z) = \frac{\phi_1(z)}{z-3i}$ or $\frac{\phi_2(z)}{z+3i}$

$$\phi_1 = \frac{z+1}{z+3i} \qquad \phi_2 = \frac{z+1}{z-3i}$$

(b) $f(z) = \frac{\phi(z)}{z-1}$, $\phi(z) = z^2+2$

(c) $f(z) = \frac{\phi(z)}{(z+\frac{1}{2})^3}$, $\phi(z) = \frac{z^3}{8}$

(d) $f(z) = \frac{\phi_1(z)}{z-\pi i}$ or $\frac{\phi_2(z)}{z+\pi i}$,

$$\phi_1 = \frac{e^z}{z+\pi i} \qquad \phi_2 = \frac{e^z}{z-\pi i}$$

3. (a) $f(z) = \frac{1}{z^4}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!} = \frac{1}{z^3} + \frac{1}{6z} + \frac{z}{120} + \cdots$

(b) $f(z) = \frac{1}{z\cdot\sum_{n=1}^{\infty}\frac{z^n}{n!}} = \frac{1}{z^2(1+\frac{z}{2}+\frac{z^2}{6}+\cdots)} = \frac{1}{z^2(1+R(z))}$

For $|z|\le 1$, we have $|R(z)|<1$, and

$$\frac{1}{1+R(z)} = 1 - R(z) + R(z)^2 - R(z)^3 + \cdots \quad \text{(will converge)}$$

We only need to plug in $1-R(z)$ to obtain the principal part, which is $\frac{1}{z^2} - \frac{1}{2z}$

P. 253: 8

The suggestion has already provided a step-by-step sketch of proof.